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THE SLOW MOTION OF TWO TOUCHING FLUID SPHERES ALONG THEIR LINE OF CENTERS

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Summary

The creeping motion along their line of centers of two fluid spheres in contact is analyzed. An exact solution is presented. Corrections to the Hadamard–Rybczynski equation are tabulated for various particle radii ratios and particle fluid to external fluid viscosity ratios. In the limit of infinite particle viscosity, these corrections are shown to agree with previous calculations for rigid spheres.

Introduction

The fluid motions produced by the migration of a bubble or droplet through a quiescent viscous fluid have been the subject of numerous investigations. The creeping flow translation of a single fluid sphere was analyzed by Hadamard [1] and independently by Rybczynski [2]. Assuming continuous tangential velocity and continuous tangential shearing stress across the interface of the phases, they found that the force exerted on the sphere by the continuous fluid phase is

$$F^z = -6\pi\mu^e a_s U \frac{3\sigma + 2}{3\sigma + 3} \tag{1.1}$$

In practice, the observed drag on a bubble is often much closer to the Stokes drag on a rigid sphere. Additionally, the internal circulation of these bubbles is not as strong as predicted by the Hadamard–Rybczynski model. The explanation is that the continuous liquid phase normally contains surface-active substances (surfactants) which are transported to the bubble surface and accumulated there. Because of the tangential surface velocity associated with internal circulation, these surfactants are carried to the rear of the bubble. The concentration of surfactants is consequently greater at the rear than at the bubble front where new surface is continuously being formed. Because of this higher concentration of surfactants at the rear, the surface tension is lower there. A net force on the surface due to the surface tension gradient results. This force opposes the surface motion causing the bubble to move with reduced internal circulation, more nearly like a rigid sphere. The earlier work on the effect of surfactants is described by Levich [3]. These earlier models, however, do not describe the cap of rigid surface observed at the rear of the bubble by Savic [4]. These observations reveal that the fluid motion is not simply a slower motion with fore and aft symmetry. Instead, the circulation and surrounding flow is asymmetric. The rear of the bubble is a rigid spherical cap resulting from the surface convection of relatively insoluble surfactant. The remainder of the bubble surface has negligibly small tangential stress because of the low concentration of surfactant and the negligible internal viscosity.

Davis and Acrivos [5] have analyzed the motion of bubbles with stagnant caps. Their analysis, based on the observations of Savic, yields a bubble drag dependence in agreement with the experimental results of Bond and Newton [6]. These results demonstrate that for sufficiently large bubbles, the drag is given by the Hadamard–Rybczynski law rather than Stokes law. They further demonstrate that under these conditions, the cap angle is small.

The purpose of this paper is to extend the Hadamard–Rybczynski law to describe the motion of a pair of contacting fluid spheres moving along their line of centers. The results of this analysis are subject to the same restrictions implicit in the use of the Hadamard–Rybczynski law. Additionally, one must impose restrictions necessary to ensure that the film separating the spheres does not immediately fail resulting in their coalescence. The motion of fluid spheres in apparent contact has been frequently observed with drops, *e.g.*, by Bartok and Mason [7] and Mackay and Mason [8] and bubbles, *e.g.*, by Pattle [9] and Soo [10]. The rate of film thinning between the fluid spheres depends on several factors. The presence of surfactant on a bubble surface retards the film thinning between bubbles and hinders coalescence. Because surfactant on the surface of the leading bubble is swept back toward the apparent contact region, a region of relatively stagnant fluid, trace amounts of surfactant may delay coalescence without significantly affecting the drag.

The hydrodynamic interaction of a fluid sphere with a second fluid sphere or with a plane interface has been the subject of previous investigations. Bart [11] examined the motion of a fluid sphere, immersed in a second fluid, moving normal to a plane interface bounding yet another fluid. Wacholder and Weihs [12] analyzed the motion of a fluid sphere through another fluid normal to a rigid or free plane surface. Wacholder and Weihs' calculations agree with the results obtained by Bart in these limits. Wacholder and Weihs also treated the case of two equal-sized and separated spheres moving with equal velocities along their line of centers. This final case is an extension of the analysis of Stimson and Jeffery [13], who analyzed the fluid motion generated by two rigid spheres moving parallel to their line of centers with equal velocities. A bispherical coordinate system was used in each of these analyses and exact solutions of the creeping flow equations obtained. The continuous tangential velocity and continuous tangential stress boundary conditions were employed at each interface between fluid phases.

While the solutions obtained using the bispherical coordinate system are exact and valid for any small separation, these solutions are not suitable for examining the motion of two spheres when the separation vanishes. The number of terms in the series solution that must be retained continues to grow without bound as the separation between the spheres decreases. The motion of a doublet of equal rigid spheres translating along its axis was first treated by Faxen [14], who examined the limit of the Stimson and Jeffery force expression as the separation between the spheres vanished. Recently, this motion has been reexamined and extended by Cooley and O'Neill [15] and independently by Goren [16]. These recent works utilize the tangent sphere coordinate system to simplify writing the boundary conditions at the sphere surfaces. The analyses yielded exact solutions of the creeping flow equations. Results were obtained for the drag on doublets of unequal rigid spheres as well as the case treated by Faxen. Yet another method of treating slow viscous flow past axisymmetric assemblages of rigid particles is presented by Gluckman *et al.* [17]. Their multipole truncation technique is a rapidly converging scheme suitable even for small spacings or touching bodies.

We intend here to extend the analysis of Wacholder and Weihs to include the motion of two fluid spheres in apparent contact translating along their line of centers. By using a tangent sphere coordinate system, results are readily obtained for pairs of arbitrary sized spheres.

Analysis of the fluid motion

Consider the motion of two touching fluid spheres translating along their line of centers. The motion of each of the fluids is governed by the creeping

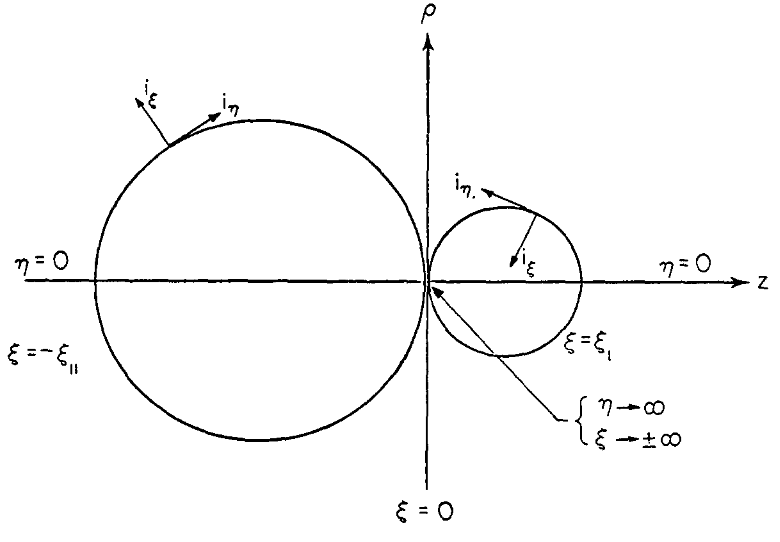

Fig. 1. Tangent sphere coordinates.

flow equations subject to conditions imposed at the boundaries. The shape of the boundary between fluid phases suggests the use of the tangent sphere coordinate system. This coordinate system is shown in Fig. 1. The tangent sphere coordinate system (η, ξ, ϕ) is a rotational orthogonal curvilinear coordinate system related to cylindrical coordinates (z, ρ, ϕ) by

$$z = \frac{\xi}{\xi^2 + \eta^2}\,; \qquad \rho = \frac{\eta}{\xi^2 + \eta^2}\,; \qquad \phi = \phi \tag{2.1}$$

The scale factors or metric coefficients are

$$h_\xi = h_\eta = \frac{1}{\xi^2 + \eta^2}\,; \qquad h_\phi = \rho = \frac{\eta}{\xi^2 + \eta^2} \tag{2.2}$$

In this coordinate system, a sphere of radius $|2\xi|^{-1}$ with its center located on the axis at $z = (2\xi)^{-1}$ is the constant ξ coordinate surface. The constant η coordinate surfaces are tori with circular cross-sections of radius $(2\eta)^{-1}$ centered at $z = 0$ and $\rho = (2\eta)^{-1}$. Tangent sphere coordinates are described in great detail by Moon and Spencer [18].

The governing equation for axisymmetric creeping flow is

$$E^4 \psi = 0 \tag{2.3}$$

where ψ is the Stokes stream function and the Stokes stream function operator, E^2, is given by

$$E^2 = \eta(\xi^2 + \eta^2) \left[\frac{\partial}{\partial \xi} \left(\frac{\xi^2 + \eta^2}{\eta} \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{\xi^2 + \eta^2}{\eta} \frac{\partial}{\partial \eta} \right) \right] \tag{2.4}$$

in tangent sphere coordinates.

A reference frame is fixed to the spheres so that there is no relative normal fluid velocity at either surface. In terms of the Stokes stream function, this boundary condition becomes

$$\psi^e = 0 \quad \text{at} \quad \xi = \xi_I \tag{2.5}$$

and

$$\psi^e = 0 \quad \text{at} \quad \xi = -\xi_{II} \tag{2.6}$$

for the continuous phase, and

$$\psi^i_I = 0 \quad \text{at} \quad \xi = \xi_I \tag{2.7}$$

and

$$\psi^i_{II} = 0 \quad \text{at} \quad \xi = -\xi_{II} \tag{2.8}$$

for the fluids internal to the spheres. The subscripts I and II refer to the spheres defined by the ξ_I and $-\xi_{II}$ coordinate surfaces.

The remaining boundary conditions imposed at the fluid interfaces are the Hadamard—Rybczynski conditions of continuous tangential velocity and continuous tangential stress across the interfaces. These conditions provide the coupling between the inner and outer flow fields. The continuous tangential velocity condition is written in terms of the stream function as

$$\frac{\partial \psi^{e}}{\partial \xi} = \frac{\partial \psi^{i}_{I}}{\partial \xi} \quad \text{at} \quad \xi = \xi_{I} \tag{2.9}$$

and

$$\frac{\partial \psi^{e}}{\partial \xi} = \frac{\partial \psi^{i}_{II}}{\partial \xi} \quad \text{at} \quad \xi = -\xi_{II} \tag{2.10}$$

in the frame of the spheres. The continuous tangential stress condition is

$$\Pi^{e}_{\xi\eta} = \Pi^{i}_{\xi\eta} \quad \text{at} \quad \xi = \xi_{I} \tag{2.11}$$

and

$$\Pi^{e}_{\xi\eta} = \Pi^{i}_{\xi\eta} \quad \text{at} \quad \xi = -\xi_{II} \tag{2.12}$$

Far from the fluid spheres, the flow is uniform streaming in the negative z-direction. The stream function of the exterior fluid accordingly has the limit

$$\psi \to \frac{U\rho^{2}}{2} = \frac{U\eta^{2}}{2(\xi^{2}+\eta^{2})^{2}} \tag{2.13}$$

as the distance from the doublet increases without bound. This limit corresponds to $\xi \to 0$, $\eta \to 0$.

A solution of (2.3) satisfying (2.5) through (2.13) is sought. Like Laplace's equation, the equation

$$E^{2}\psi = 0 \tag{2.14}$$

is R-separable in tangent sphere coordinates. Additionally, solutions to the governing equation (2.3) are readily found from solutions of (2.14). If we let

$$\psi = (\xi^{2}+\eta^{2})^{-1/2}\, G(\xi,\eta) \tag{2.15}$$

and substitute into (2.14), the resulting equation for G separates into ordinary differential equations in ξ and η.

$$G = N(\xi)\, M(\eta) \tag{2.16}$$

$$\frac{d^{2}N}{d\xi^{2}} - \lambda^{2} N = 0 \tag{2.17}$$

$$\eta^{2}\frac{d^{2}M}{d\eta^{2}} - \eta\frac{dM}{d\eta} + \lambda^{2}\eta^{2} M = 0 \tag{2.18}$$

The solution of the linear differential equation with constant coefficients (2.17) is

$$N = c_{1}\sinh \lambda\xi + c_{2}\cosh \lambda\xi \tag{2.19}$$

while the solution of (2.18), a form of Bessel's equation, is

$$M = \eta [c_3 J_1(\lambda\eta) + c_4 Y_1(\lambda\eta)] \tag{2.20}$$

$J_1(\lambda\eta)$ and $Y_1(\lambda\eta)$ are Bessel's functions of the first and second kind and order unity. Since $Y_1(\lambda\eta)$ is unbounded at $\eta = 0$, this solution is omitted and the solution of (2.14) is

$$\psi = (\xi^2 + \eta^2)^{-1/2} \eta \int_0^\infty J_1(\lambda\eta)[c_1(\lambda) \sinh \lambda\xi + c_2(\lambda) \cosh \lambda\xi] \, d\lambda \tag{2.21}$$

Majumdar [19] obtained the solution of (2.14) in his analysis of the axisymmetric irrotational flow past two touching spheres.

Any solution of (2.14) is a solution of (2.3). Stimson and Jeffery have shown that if a function ψ satisfies (2.14), then $z\psi$ also satisfies (2.3). Any linear combination of these solutions of (2.3) is also a solution. Accordingly, (2.21) can be used to generate a sufficiently general solution of the creeping flow equation for our purposes.

For the external fluid, the solution is, using (2.13), written

$$\psi^{\mathrm{e}} = U\eta (\xi^2 + \eta^2)^{-3/2} \int_0^\infty F^{\mathrm{e}}(\xi) J_1(\lambda\eta) \, d\lambda + \frac{U\eta^2}{2(\xi^2 + \eta^2)^2} \tag{2.22}$$

where

$$F^{\mathrm{e}}(\xi) = (A + C\xi) \sinh \lambda\xi + (B + D\xi) \cosh \lambda\xi \tag{2.23}$$

and A, B, C, D are functions of λ. Bounded solutions in the interiors of the fluid spheres are

$$\psi^{\mathrm{i}}_{\mathrm{I,II}} = U\eta(\xi^2 + \eta^2)^{-3/2} \int_0^\infty F^{\mathrm{i}}_{\mathrm{I,II}} J_1(\lambda\eta) \, d\lambda \tag{2.24}$$

with

$$F^{\mathrm{i}}_{\mathrm{I}} = (a + b\xi) \, e^{-\lambda\xi} \tag{2.25}$$

and

$$F^{\mathrm{i}}_{\mathrm{II}} = (c + d\xi) \, e^{\lambda\xi} \tag{2.26}$$

$a, b, c,$ and d are also functions of λ. The eight coefficients are found by simultaneously satisfying the eight boundary conditions (2.5 through 2.12) imposed at the fluid sphere surfaces. The following relations, obtained using tabulated Hankel transforms [20], are useful in writing the boundary conditions

$$\eta(\xi^2 + \eta^2)^{-1/2} = \int_0^\infty e^{-\lambda|\xi|} (|\xi| + \lambda^{-1}) J_1(\lambda\eta) \, d\lambda \tag{2.27}$$

$$\frac{d}{d\xi}\left[\eta(\xi^2+\eta^2)^{-1/2}\right] = -\xi \int_0^\infty \lambda\, e^{-\lambda|\xi|} J_1(\lambda\eta)\, d\lambda \tag{2.28}$$

$$\frac{d^2}{d\xi^2}\left[\eta(\xi^2+\eta^2)^{-1/2}\right] = -\int_0^\infty \lambda(1-\lambda|\xi|)\, e^{-\lambda|\xi|} J_1(\lambda\eta)\, d\lambda \tag{2.29}$$

Now, the no normal flow boundary conditions for the external fluid (2.5 and 2.6) become

$$A \sinh \lambda\xi_I + B \cosh \lambda\xi_I + C\xi_I \sinh \lambda\xi_I + D\xi_I \cosh \lambda\xi_I = -e^{-\lambda\xi_I}(\xi_I + \lambda^{-1})/2 \tag{2.30}$$

at the surface of sphere I and

$$-A \sinh \lambda\xi_{II} + B \cosh \lambda\xi_{II} + C\xi_{II} \sinh \lambda\xi_{II} - D\xi_{II} \cosh \lambda\xi_{II} = -e^{-\lambda\xi_{II}}(\xi_{II} + \lambda^{-1})/2 \tag{2.31}$$

at the surface of sphere II. Similarly, for the interior fluid, the conditions (2.7 and 2.8) become

$$a + \xi_I b = 0 \tag{2.32}$$

and

$$c - \xi_{II} d = 0 \tag{2.33}$$

The tangential velocity boundary conditions (2.9 and 2.10) become

$$\begin{aligned} &-\lambda a\, e^{-\lambda\xi_I} + b\, e^{-\lambda\xi_I}(1-\lambda\xi_I) - A\lambda \cosh \lambda\xi_I - B\lambda \sinh \lambda\xi_I \\ &\quad -C(\lambda\xi_I \cosh \lambda\xi_I + \sinh \lambda\xi_I) - D(\lambda\xi_I \sinh \lambda\xi_I + \cosh \lambda\xi_I) \\ &\quad = -\lambda\xi_I\, e^{-\lambda\xi_I}/2 \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} &\lambda c\, e^{-\lambda\xi_{II}} + d\, e^{-\lambda\xi_{II}}(1-\lambda\xi_{II}) - A\lambda \cosh \lambda\xi_{II} + B\lambda \sinh \lambda\xi_{II} \\ &\quad +C(\lambda\xi_{II} \cosh \lambda\xi_{II} + \sinh \lambda\xi_{II}) - D(\lambda\xi_{II} \sinh \lambda\xi_{II} + \cosh \lambda\xi_{II}) \\ &\quad = \lambda\xi_{II}\, e^{-\lambda\xi_{II}}/2 \end{aligned} \tag{2.35}$$

The tangential stress component $\Pi_{\xi\eta}$ is written in terms of the stream function in tangent sphere coordinates as

$$\Pi_{\xi\eta} = \mu\eta^{-1}(\xi^2+\eta^2)^2 \left[(\xi^2+\eta^2)\left(\frac{\partial^2\psi}{\partial\eta^2} - \frac{1}{\eta}\frac{\partial\psi}{\partial\eta} - \frac{\partial^2\psi}{\partial\xi^2}\right) - 6\xi\frac{\partial\psi}{\partial\xi} + 6\eta\frac{\partial\psi}{\partial\eta}\right] \tag{2.36}$$

which, using (2.5) through (2.8), reduces to

$$\Pi_{\xi\eta} = -\mu\eta^{-1}(\xi^2+\eta^2)^{3/2} \frac{\partial^2}{\partial\xi^2} \left[(\xi^2+\eta^2)^{3/2} \psi\right] \tag{2.37}$$

at the surfaces of the fluid spheres. The tangential stress boundary conditions (2.11 and 2.12) become

$$\sigma_{\mathrm{I,II}} \frac{\partial^2}{\partial\xi^2} \left[(\xi^2+\eta^2)^{3/2} \psi^{\mathrm{i}}_{\mathrm{I,II}}\right] = \frac{\partial^2}{\partial\xi^2} \left[(\xi^2+\eta^2)^{3/2} \psi^{\mathrm{e}}\right] \tag{2.38}$$

at the two surfaces. σ refers to the viscosity ratio.

$$\sigma_{\mathrm{I,II}} = \mu^{\mathrm{i}}_{\mathrm{I,II}}/\mu^{\mathrm{e}} \tag{2.39}$$

In terms of the coefficients in the stream function expression, the continuous tangential stress boundary conditions are

$$\begin{aligned}\sigma_{\mathrm{I}} &\left[\lambda a\, e^{-\lambda\xi_{\mathrm{I}}} - b\, e^{-\lambda\xi_{\mathrm{I}}}(2-\lambda\xi_{\mathrm{I}})\right] - A\lambda \sinh\lambda\xi_{\mathrm{I}} - B\lambda\cosh\lambda\xi_{\mathrm{I}} \\ &- C(2\cosh\lambda\xi_{\mathrm{I}} + \lambda\xi_{\mathrm{I}}\sinh\lambda\xi_{\mathrm{I}}) - D(2\sinh\lambda\xi_{\mathrm{I}} + \lambda\xi_{\mathrm{I}}\cosh\lambda\xi_{\mathrm{I}}) \\ &= (\lambda\xi_{\mathrm{I}}-1)\, e^{-\lambda\xi_{\mathrm{I}}}/2\end{aligned} \tag{2.40}$$

and

$$\begin{aligned}\sigma_{\mathrm{II}} &\left[\lambda c\, e^{-\lambda\xi_{\mathrm{II}}} + d\, e^{-\lambda\xi_{\mathrm{II}}}(2-\lambda\xi_{\mathrm{II}})\right] + A\lambda\sinh\lambda\xi_{\mathrm{II}} \\ &- B\lambda\cosh\lambda\xi_{\mathrm{II}} - C(2\cosh\lambda\xi_{\mathrm{II}} + \lambda\xi_{\mathrm{II}}\sinh\lambda\xi_{\mathrm{II}}) \\ &+ D(2\sinh\lambda\xi_{\mathrm{II}} + \lambda\xi_{\mathrm{II}}\cosh\lambda\xi_{\mathrm{II}}) = (\lambda\xi_{\mathrm{II}}-1)\, e^{-\lambda\xi_{\mathrm{II}}}/2\end{aligned} \tag{2.41}$$

Simultaneous solution of the eight equations, (2.30) through (2.35), (2.40) and (2.41) yields values of the eight coefficients in terms of λ, the sphere radius ratio and the viscosity ratios.

Forces on the spheres

Cooley and O'Neill as well as Goren used an expression due to Stimson and Jeffery to find the forces exerted by the surrounding fluid on the two rigid spheres. In a steady axisymmetric creeping flow, the force exerted on a body by the fluid is the axial force

$$F^z = \pi\mu \int \frac{\rho^3}{h_1} \frac{\partial}{\partial u_1} \left(\frac{E^2\psi}{\rho^2}\right) h_2 \, du_2 \tag{3.1}$$

While Stimson and Jeffery applied a no-slip condition at the body surface to obtain this result, it has been noted [21, 22] that the no-slip condition may be removed. The result is equally valid for flow past fluid spheres where the tangential velocity of the fluids does not vanish at the interface. Accordingly, the tangent sphere formulations of this expression of Cooley and O'Neill and of Goren are equally applicable here. In terms of the stream function coefficients, the forces on the fluid spheres are

$$F^z_{\mathrm{I,II}} = -4\pi\mu^e a_{\mathrm{I,II}} U \int_0^\infty \lambda(B \pm A)\, d\lambda \tag{3.2}$$

where the upper sign refers to sphere I and the lower sign refers to sphere II. The force on a sphere is more usefully written in terms of the force on an isolated fluid sphere (1.1) and a correction, β, to account for the effect of the second sphere's presence.

$$F^z_{\mathrm{I,II}} = -6\pi\mu^e a_{\mathrm{I,II}} U \frac{3\sigma_{\mathrm{I,II}} + 2}{3\sigma_{\mathrm{I,II}} + 3} \beta_{\mathrm{I,II}} \tag{3.3}$$

$$\beta_{\mathrm{I,II}} = \frac{2\sigma_{\mathrm{I,II}} + 2}{3\sigma_{\mathrm{I,II}} + 2} \int_0^\infty \lambda(B \pm A)\, d\lambda \tag{3.4}$$

The factor, β, was calculated for a fluid sphere, say sphere I, under a variety of conditions. For selected values of the radius ratio, $R \equiv a_{\mathrm{I}}/a_{\mathrm{II}}$, and selected equal viscosity ratios, $\sigma \equiv \sigma_{\mathrm{I}} = \sigma_{\mathrm{II}}$, equations (2.30) through (2.35), (2.40), and (2.41) were solved numerically yielding the coefficients as functions of λ. The integral of equation (3.4) was then evaluated numerically using Legendre–Gauss quadrature and the resulting values of β presented in Table 1.

TABLE 1

Correction, β, to drag on a fluid sphere resulting from adjacent sphere

R	$\sigma = 0.0$	$\sigma = 0.5$	$\sigma = 1.0$	$\sigma = 5.0$	$\sigma = 10.0$	$\sigma = 100$	$\sigma = 1000$	$\sigma = 10^6$
0.10	0.14579	0.12535	0.11250	0.07619	0.06327	0.04170	0.03716	0.03637
0.25	0.31023	0.27945	0.26049	0.20874	0.19101	0.16228	0.15641	0.15539
0.50	0.49401	0.46516	0.44806	0.40447	0.39063	0.36948	0.36541	0.36471
1.00	0.69315	0.67783	0.66971	0.65327	0.64956	0.64562	0.64519	0.64514
2.00	0.85161	0.84965	0.84963	0.85448	0.85768	0.86435	0.86597	0.86627
4.00	0.94236	0.94479	0.94687	0.95486	0.95829	0.96447	0.96584	0.96608
10.00	0.98724	0.98861	0.98958	0.99268	0.99388	0.99596	0.99641	0.99648

For equal-sized spheres, $R = 1$, and for any viscosity ratio, β should be slightly smaller than the corresponding factor presented by Wacholder and Weihs at the closest separation calculated by them (distance between centers/diameter = 1.005). In all cases, β satisfies this check. In the limit of infinite viscosity ratio, the spheres are rigid. Our calculations for $\sigma = 10^6$ agree with the rigid sphere results of Goren and of Cooley and O'Neill, at each value of R.

For equal-sized spheres with identical viscosity ratios, the coefficients of the stream functions have been determined algebraically. Choosing $\xi_{I} = \xi_{II} = 1$, the correction to the Hadamard—Rybczynski law is

$$\beta = \frac{\sigma + 1}{3\sigma + 2} \int_0^\infty \frac{e^{-\lambda}\{[\sigma(\lambda^2 + \lambda + 1) + \lambda] \sinh \lambda + (\lambda + 1)(\sigma\lambda + 1) \cosh \lambda\}}{\sigma(\lambda + \cosh \lambda \sinh \lambda) + \cosh^2 \lambda} d\lambda$$

For infinite viscosity ratio, rigid spheres, this expression reduces to the result of Faxen

$$\beta = \frac{1}{3} \int_0^\infty \left(1 - \frac{2 \sinh^2 \lambda - 2\lambda^2}{\sinh 2\lambda + 2\lambda}\right) d\lambda$$

while for a zero viscosity ratio, approximating bubble motion through a liquid,

$$\beta = \frac{1}{2} \int_0^\infty \frac{e^{-\lambda} [\lambda \sinh \lambda + (\lambda + 1) \cosh \lambda]}{\cosh^2 \lambda} d\lambda$$

Conclusions

The effect of internal circulation on the hydrodynamic interaction of two touching fluid spheres migrating along their line of centers has been investigated. Exact solutions were obtained and the correction to the drag on a sphere was tabulated for several radius ratios and viscosity ratios.

For equal-sized spheres, the effect of greater internal circulation, corresponding to smaller dispersed phase viscosity, is to decrease the effect on the drag caused by the adjacent sphere. This effect was also obtained by Wacholder and Weihs in the analysis of separated fluid spheres.

For spheres of widely disparate sizes, the effect is more subtle. In the case of the smaller sphere, smaller dispersed phase viscosity again serves to decrease the influence of the larger sphere on the drag. However, the smaller viscosity also increases the smaller sphere's influence on the larger sphere's drag. Each of these effects is readily explained in terms of the stagnant fluid region fore and aft of the larger sphere. As the sphere viscosity increases, the extent of stagnation about the smaller sphere increases producing these results. The same behavior can be anticipated in the interaction of separated fluid spheres of significantly different size.

Nomenclature

A,B,C,D	stream function coefficients defined by (2.23)
a,b,c,d	stream function coefficients defined by (2.25) and (2.26)
a_s,a_I,a_{II}	sphere radii
E^2	Stokes stream function operator
F^z	hydrodynamic force on sphere
$h_\xi,h_\eta,h_\phi,h_1,h_2$	metric coefficients
R	radius ratio, a_I/a_{II}
U	velocity of spheres
u_1,u_2	orthogonal curvilinear coordinates
z	axial position in cylindrical coordinates

Greek symbols

β	correction to Hadamard—Rybczynski law accounting for effect of neighboring sphere
η	tangent sphere coordinate
λ	separation variable
μ	viscosity
ξ	tangent sphere coordinate
Π	stress component
ρ	cylindrical radial position
σ	viscosity ratio defined by (2.39)
ψ	Stokes stream function

Superscripts

e	external fluid
i	internal fluid

Subscripts

I	sphere I
II	sphere II

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